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Dielectric Formulation of Test Particle Energy Loss in a Plasma

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Using a perturbation-theoretic method which starts from a microscopic Newtonian equation of motion for the trajectory of a test particle moving in a magnetic field-free plasma, the polarization and statistical contributions to the test charge energy loss are formulated entirely in terms of linear and quadratic dielectric functions.

KEY WORDS: Energy loss; power loss; test particle; nonlinear correction; dielectric description; fluctuation-dissipation theorem; magnetic field-free plasma.

1. INTRODUCTION

There are two principal mechanisms which scatter a test particle in a plasma. The first mechanism relates to the polarization of the medium and the second to its statistical fluctuations.

Concerning polarization, when the test charge enters the plasma a screening cloud forms about the test charge, exerting a net drag force on it. The interactions are brought about both by the collective $(r > L_d)$ behavior of the plasma particles and by many-body test-charge field-particle interactions $(r < L_d)$. Kronig and Koringa, Kramers, and Neufeld and Ritchie[1] first analyzed the extent of this polarization by considering the plasma to be a continuous polarizable fluid; the presence of the test charge then brings about an average electric field response which, in turn, decelerates the test charge. Chandrasekhar and others[2] also considered the polarization aspect

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of scattering using test-charge–plasma-particle binary collision models. However, the long-range character of plasma interactions renders binary collision models less suitable than the self-consistent formulations of the previous authors cited.[1]

Concerning the second principal scattering mechanism, the test particle is heated up by the thermal fluctuations of the plasma particles. Gasiorowicz *et al.*[3] were able to take account of statistical effects in a Fokker–Planck model which viewed the plasma system as a source of random microforces acting on the test charge. The most systematic treatment, however, which starts from a microscopic Newtonian equation of motion for the test charge in a classical plasma, was presented by Kalman and Ron.[4] Their perturbation-theoretic study, which features the coupling strength e^2 (which, in dimensionless form is the ratio of the potential energy to thermal energy) as the smallness parameter, yields the test charge energy loss expressions for the pure polarization and pure statistical effects (these occur to lowest order e^2) and for the mixed statistical-polarization effect (occurs to e^4 lowest order). Kalman[5] later reformulated the $O(e^2)$ energy loss contributions in terms of the *linear* wave-vector- and frequency-dependent dielectric function (along the lines of the fast electron energy loss calculation discussed in the quantum liquid studies of Pines and Nozieres[6]) by making use of the statistical mechanical fluctuation-dissipation theorem.[7] This elegant description of energy loss not only provides a deeper insight into the structure of the Fokker–Planck coefficients, but enables one to determine the energy loss in terms of the *Vlasov* expression for the linear polarizability. Indeed, one could calculate the linear polarizability from a kinetic equation for the one-particle distribution function which displays a collision operator on its r.h.s. We shall see, however, that the subsequent introduction of such higher-order collisional corrections must, for the sake of completeness, be accompanied by the inclusion of the Vlasov expression for the quadratic polarizability[8] in a dielectric formulation of energy loss.

Recently we were able to establish a nonlinear fluctuation-dissipation theorem providing the connection between a single dynamical equilibrium triplet correlation of microscopic current (or charge) densities and a combination of three wave-vector- and frequency-dependent quadratic polarizabilities.[9] The development of this theorem therefore makes it possible to extend Kalman's dielectric formulation to higher than first order in the coupling strength. This is the main purpose of the present paper.

This paper is divided into five sections. In Section 2 the perturbation scheme is laid out, featuring the charge of the test particle as the smallness parameter. For the sake of mathematical simplicity, we restrict ourselves here to a consideration of magnetic field-free plasmas. In Section 3 we use medium electrodynamics to develop the linear and quadratic expressions for the power loss due only to the pure polarization effects. In Section 4 all of the correlations of the microscopic electric fields, which are enumerated in Section 2, are recast into *equilibrium* ensemble-averaged correlations of microscopic current densities suitable for conversion in Section 5 to linear and quadratic polarizability functions. Finally, in Section 6 the power loss expressions are shown, and a Vlasov calculation for the quadratic polarizability is displayed in order to properly feature the coupling strength e^2 as the smallness parameter, thereby bringing our expansion scheme into line with the customary scheme of plasma physics.

2. PERTURBATION SCHEME

In the absence of plasma and externally applied magnetic fields, the moving test particle (of mass m_0 and charge Ze) traces out a straight line path at constant velocity $\mathbf{v}_0$. At $t = 0$ we introduce the test charge into a classical magnetic field-free plasma. Then for $t > 0$ the total microscopic electric field $\mathbf{E}(\mathbf{x}(t), t)$ of the plasma particles brings about the small microscopic change $\Delta\mathbf{v}$ in the velocity of the test charge. The microscopic equation of motion,

$$(d/dt)\,\Delta\mathbf{v}(t) = (Ze/m_0)\,\mathbf{E}(t) \tag{1}$$

shows that, to lowest order, the smallness of $\Delta\mathbf{v}$ corresponds to the assumed smallness of the test charge. Its position $\mathbf{x}(t)$ is evidently given by

$$\mathbf{x}(t) = \mathbf{x}(0) + \mathbf{v}_0 t + \Delta\mathbf{x}(t) \tag{2}$$

where

$$\Delta\mathbf{x}(t) = \int_0^t dt'\,\Delta\mathbf{v}(t')$$

Without loss of generality, one can set $\mathbf{x}(0) = 0$. Next, we Taylor-expand the microscopic field

$$E_\mu(\mathbf{x}(t), t) = E_\mu(\mathbf{v}_0 t, t) + E_{\mu,\nu}(\mathbf{v}_0 t, t)\,\Delta x_\nu(t) + \tfrac{1}{2}E_{\mu,\nu\lambda}(\mathbf{v}_0 t, t)\,\Delta x_\nu(t)\,\Delta x_\lambda(t) \tag{3}$$

about the unperturbed orbit $\mathbf{v}_0 t$ of the test particle and set

$$\Delta v_\mu(t) = \Delta v_\mu^{(1)}(t) + \Delta v_\mu^{(2)}(t) + \cdots \tag{4a}$$

$$\Delta x_\mu(t) = \Delta x_\mu^{(1)}(t) + \Delta x_\mu^{(2)}(t) + \cdots \tag{4b}$$

where the ith superscript denotes, for example, that the correction $\Delta v_\mu^{(i)}$ is of $O[(Ze)^i]$ in smallness.

The application of the expansions (3) and (4) to (1) then gives the equations of motion:

$$(d/dt)\,\Delta v_\mu^{(1)}(t) = (Ze/m_0)\,E_\mu(\mathbf{v}_0 t, t) \tag{5a}$$

$$(d/dt)\,\Delta v_\mu^{(2)}(t) = (Ze/m_0)\,E_{\mu,\nu}(\mathbf{v}_0 t, t)\,\Delta x_\nu^{(1)}(t) \tag{5b}$$

$$\begin{aligned}(d/dt)\,\Delta v_\mu^{(3)}(t) = {} & (Ze/m_0)[E_{\mu,\nu}(\mathbf{v}_0 t, t)\,\Delta x_\nu^{(2)}(t) \\ & + \tfrac{1}{2}E_{\mu,\nu\lambda}(\mathbf{v}_0 t, t)\,\Delta x_\nu^{(1)}(t)\,\Delta x_\lambda^{(1)}(t)] \cdots\end{aligned} \tag{5c}$$

with subsequent integrals

$$\Delta v_\mu^{(1)}(t) = (Ze/m_0)\int_0^t dt'\; E_\mu(\mathbf{v}_0 t', t') \tag{6a}$$

$$\Delta v_\mu^{(2)}(t) = (Ze/m_0)^2 \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt'''\; E_{\mu,\nu}(\mathbf{v}_0 t', t')\, E_\nu(\mathbf{v}_0 t''', t''') \tag{6b}$$

$$\begin{aligned}\Delta v_\mu^{(3)}(t) = {} & (Ze/m_0)^3 \Big\{ \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' \int_0^{t'''} dt^{\mathrm{IV}} \int_0^{t^{\mathrm{IV}}} dt^{\mathrm{V}} \\ & \times E_{\mu,\nu}(\mathbf{v}_0 t', t')\, E_{\nu,\lambda}(\mathbf{v}_0 t''', t''')\, E_\lambda(\mathbf{v}_0 t^{\mathrm{V}}, t^{\mathrm{V}}) \\ & + \frac{1}{2}\int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' \int_0^{t'} dt^{\mathrm{IV}} \int_0^{t^{\mathrm{IV}}} dt^{\mathrm{V}} \\ & \times E_{\mu,\nu\lambda}(\mathbf{v}_0 t', t')\, E_\nu(\mathbf{v}_0 t''', t''')\, E_\lambda(\mathbf{v}_0 t^{\mathrm{V}}, t^{\mathrm{V}}) \Big\}\end{aligned} \tag{6c}$$

$$\vdots$$

Next, we turn our attention to a calculation of the test charge energy loss,

$$\begin{aligned}\Delta T &= \tfrac{1}{2}m_0 v_0^2 - \tfrac{1}{2}m_0(\mathbf{v}_0 + \Delta\mathbf{v})\cdot(\mathbf{v}_0 + \Delta\mathbf{v}) \\ &= -m_0\mathbf{v}_0\cdot\Delta\mathbf{v}(t) - \tfrac{1}{2}m_0\Delta\mathbf{v}(t)\cdot\Delta\mathbf{v}(t)\end{aligned} \tag{7}$$

which, from (4a), can be decomposed into the corrections

$$\Delta T^{(1)}(t) = -m_0 v_{0\mu}\Delta v_\mu^{(1)}(t) \tag{8a}$$

$$\Delta T^{(2)}(t) = -m_0 v_{0\mu}\Delta v_\mu^{(2)}(t) - \tfrac{1}{2}m_0\Delta v_\mu^{(1)}(t)\,\Delta v_\mu^{(1)}(t) \tag{8b}$$

$$\Delta T^{(3)}(t) = -m_0 v_{0\mu}\Delta v_\mu^{(3)}(t) - m_0\Delta v_\mu^{(1)}(t)\,\Delta v_\mu^{(2)}(t) \tag{8c}$$

$$\vdots$$

The ensemble-averaged energy loss[2] is then calculated according to

$$\langle \Delta T\rangle(t) = \int_{6N} \cdots \int \prod_{i=1}^{N} d^3\mathbf{x}_i\, d^3\mathbf{p}_i\, \Omega \Delta T$$
$$= \sum_{s=0}^{\infty} \int_{6N} \cdots \int \prod_{i=1}^{N} d^3\mathbf{x}_i\, d^3\mathbf{p}_i\, \Omega^{(s)} \Delta T = \sum_{s=0}^{\infty} \langle \Delta T\rangle^{(s)}(t) \tag{9}$$

where Ω is the N-particle distribution function normalized to unity, $\Omega^{(0)}$ is the macrocanonical distribution function of the unperturbed plasma particles, and $\Omega^{(s)} = O[(Ze)^s]$ is the sth correction due to the perturbing influence of the test charge. Thus [see, e.g., Eq. (18) below]

$$\langle \Delta T\rangle^{(0)} = \langle \Delta T^{(1)}\rangle^{(0)} + \langle \Delta T^{(2)}\rangle^{(0)} + \langle \Delta T^{(3)}\rangle^{(0)} + \cdots$$
$$\langle \Delta T\rangle^{(1)} = \langle \Delta T^{(1)}\rangle^{(1)} + \langle \Delta T^{(2)}\rangle^{(1)} + \cdots$$
$$\langle \Delta T\rangle^{(2)} = \langle \Delta T^{(1)}\rangle^{(2)} + \cdots$$

to third oder of smallness. Clearly $\langle \Delta T^{(1)}\rangle^{(0)} = m_0\mathbf{v}_0 \cdot \langle \Delta \mathbf{v}^{(1)}\rangle^{(0)} = 0$, since, for the equilibrium system, all possible directions of the total microscopic electric field are equally probable. We now redistribute the average energy loss into groups of like order in the driving test charge (Ze), namely

$$\langle \Delta T\rangle(t) = U^{(2)}(t) + U^{(3)}(t) + \cdots \tag{10}$$

where

$$U^{(2)}(t) = \langle \Delta T^{(1)}\rangle^{(1)} + \langle \Delta T^{(2)}\rangle^{(0)}$$
$$= m_0 v_{0\mu}\langle \Delta v_\mu^{(1)}(t)\rangle^{(1)} + m_0 v_{0\mu}\langle \Delta v_\mu^{(2)}(t)\rangle^{(0)}$$
$$+ \tfrac{1}{2} m_0 \langle \Delta v_\mu^{(1)}(t)\, \Delta v_\mu^{(1)}(t)\rangle^{(0)} \tag{11a}$$

$$U^{(3)}(t) = \langle \Delta T^{(1)}\rangle^{(2)} + \langle \Delta T^{(2)}\rangle^{(1)} + \langle \Delta T^{(3)}\rangle^{(0)}$$
$$= m_0 v_{0\mu}\langle \Delta v_\mu^{(1)}(t)\rangle^{(2)} + m_0 v_{0\mu}\langle \Delta v_\mu^{(2)}(t)\rangle^{(1)}$$
$$+ \tfrac{1}{2} m_0 \langle \Delta v_\mu^{(1)}(t)\, \Delta v_\mu^{(1)}(t)\rangle^{(1)} + m_0 \langle \Delta v_\mu^{(1)}(t)\, \Delta v_\mu^{(2)}(t)\rangle^{(0)}$$
$$+ m_0 v_{0\mu}\langle \Delta v_\mu^{(3)}(t)\rangle^{(0)} \tag{11b}$$

[2] We shall see that the average energy loss of the test particle is proportional to its residence time t in the plasma. In order that the orbit of the test charge be only slightly disturbed, its residence time must be short compared with its relaxation time. On the other hand, the notion of ensemble averaging according to (9) is meaningful only if t is sufficiently long to include many fluctuations of the plasma particles. It is physically reasonable to assume that $\tau_{\text{FLUCTUATION}} \ll \tau_{\text{RELAXATION}}$, and if the test charge interacts weakly with the equilibrium system, then $\tau_{\text{RELAXATION}} \to \infty$. Thus in the sequel we take t to be large, i.e., $t \to \infty$.

The velocity correction in (11a) and (11b) can now be eliminated in favor of the microscopic field quantities by use of (6a)–(6c). We obtain

$$U^{(2)}(t) = U^{(2)}_{\mathrm{POL}}(t) + U^{(2,1)}_{\mathrm{STAT}}(t) + U^{(2,2)}_{\mathrm{STAT}}(t) \tag{12a}$$

$$U^{(3)}(t) = U^{(3)}_{\mathrm{POL}}(t) + \sum_{s=1,2} U^{(3,s)}_{\mathrm{POL,STAT}}(t) + \sum_{s=3}^{5} U^{(3,s)}_{\mathrm{STAT}}(t) \tag{12b}$$

where

$$U^{(s+1)}_{\mathrm{POL}}(t) = Zev_{0\mu} \int_0^t d\tau \, \langle E_\mu(\mathbf{v}_0\tau, \tau)\rangle^{(s)}, \qquad s = 1, 2 \tag{13a}$$

$$U^{(2,1)}_{\mathrm{STAT}}(t) = -\frac{(Ze)^2}{m_0} v_{0\mu} \int_0^t d\tau \int_0^\tau d\tau' \int_0^{\tau'} d\tau'' \, \langle E_{\mu,\nu}(\mathbf{v}_0\tau, \tau)\, E_\nu(\mathbf{v}_0\tau', \tau')\rangle^{(0)} \tag{13b}$$

$$U^{(2,2)}_{\mathrm{STAT}}(t) = \frac{(Ze)^2}{2m_0} \int_0^t d\tau \int_0^t d\tau' \, \langle E_\mu(\mathbf{v}_0\tau, \tau)\, E_\mu(\mathbf{v}_0\tau', \tau')\rangle^{(0)} \tag{13c}$$

$$U^{(3,1)}_{\mathrm{POL,STAT}}(t) = -\frac{(Ze)^2}{m_0} v_{0\mu} \int_0^t d\tau \int_0^\tau d\tau' \int_0^{\tau'} d\tau'' \, \langle E_{\mu,\nu}(\mathbf{v}_0\tau, \tau)\, E_\nu(\mathbf{v}_0\tau'', \tau'')\rangle^{(1)} \tag{13d}$$

$$U^{(3,2)}_{\mathrm{POL,STAT}}(t) = \frac{(Ze)^2}{2m_0} \int_0^t d\tau \int_0^t d\tau' \, \langle E_\mu(\mathbf{v}_0\tau, \tau)\, E_\mu(\mathbf{v}_0\tau', \tau')\rangle^{(1)} \tag{13e}$$

$$U^{(3,3)}_{\mathrm{STAT}}(t) = \frac{(Ze)^3}{m_0^2} \int_0^t d\tau \int_0^t d\tau' \int_0^{\tau'} d\tau'' \int_0^{\tau''} d\tau''' \times \langle E_\mu(\mathbf{v}_0\tau, \tau)\, E_{\mu,\nu}(\mathbf{v}_0\tau', \tau')\, E_\nu(\mathbf{v}_0\tau''', \tau''')\rangle^{(0)} \tag{13f}$$

$$U^{(3,4)}_{\mathrm{STAT}}(t) = \frac{(Ze)^3}{m_0^2} v_{0\mu} \int_0^t d\tau \int_0^\tau d\tau' \int_0^{\tau'} d\tau'' \int_0^{\tau''} d\tau''' \int_0^{\tau'''} d\tau^{\mathrm{IV}} \times \langle E_{\mu,\nu}(\mathbf{v}_0\tau, \tau)\, E_{\nu,\lambda}(\mathbf{v}_0\tau'', \tau'')\, E_\lambda(\mathbf{v}_0\tau^{\mathrm{IV}}, \tau^{\mathrm{IV}})\rangle^{(0)} \tag{13g}$$

$$U^{(3,5)}_{\mathrm{STAT}}(t) = \frac{(Ze)^3}{2m_0^2} v_{0\mu} \int_0^t d\tau \int_0^\tau d\tau' \int_0^{\tau'} d\tau'' \int_0^{\tau} d\tau''' \int_0^{\tau'''} d\tau^{\mathrm{IV}} \times \langle E_{\mu,\nu\lambda}(\mathbf{v}_0\tau, \tau)\, E_\nu(\mathbf{v}_0\tau'', \tau'')\, E_\lambda(\mathbf{v}_0\tau^{\mathrm{IV}}, \tau^{\mathrm{IV}})\rangle^{(0)} \tag{13h}$$

The energy loss expression (12a) has already been cast into a completely *linear* dielectric form by Kalman for nonrelativistic magnetic field-free plasma systems[5] and by Golden and Kalman for relativistic plasma without externally applied magnetic fields.[10] The first r.h.s. term of (12a) arises from polarization effects [note in (13a) the perturbation in the field particle distribution function brought about by the polarizing presence of the test charge]. A glance at (13b) and (13c) shows that the last two terms of (12a) arise from statistical effects; here one can use the equilibrium fluctuation-dissipation theorem to eliminate the inconvenient correlations of the microscopic electric fields in favor of the linear dielectric function. These calculations of Kalman will be briefly reviewed in the present paper. In the expression (12b), the first r.h.s. term represents a higher-order correction to the polarization loss, while terms (13f)–(13h) are purely statistical contributions since they are simply triplet correlations averaged over the unperturbed distribution function of the plasma particles. We note, however, that (13d) and (13e) are doublet correlations averaged over the perturbed distribution of the field particles (again brought about by the polarizing influence of the moving test charge); these latter contributions therefore portray the mixed statistical-polarization effects to lowest order in (Ze).

Now the contributions (13d) and (13e) can be expressed in terms of equilibrium triplet correlations similar to (13f)–(13h) by first observing that

$$\Omega^{(1)}(t) = -i\int_0^{t} d\tau''' \,[\exp(-iL^{(0)}\tau''')]\, L^{(1)}(t-\tau''')\,\Omega^{(0)} \tag{14}$$

is a formal solution of the perturbed Liouville equation

$$(\partial\Omega^{(1)}/\partial t) + iL^{(0)}\Omega^{(1)} + iL^{(1)}\Omega^{(0)} = 0$$

where

$$L^{(s)} = i[H^{(s)}, \ldots], \qquad s = 0, 1 \tag{15}$$

are Poisson bracket operators, $H^{(0)}$ being the Hamiltonian of the unperturbed plasma including interaction, and

$$H^{(1)}(t) = \sum_{\mathbf{p}''} \hat{\phi}_{-\mathbf{p}''}(t)\rho_{\mathbf{p}''} \tag{16}$$

the Hamiltonian portraying the interaction between the external scalar potential of the moving test charge,

$$\hat{\phi}_{\mathbf{p}''}(t) = (Ze/\epsilon_0 p''^2)\exp(i\mathbf{p}''\cdot\mathbf{v}_0 t) \tag{17}$$

and the microscopic charge density $\rho_{\mathbf{p}''}$. Upon combining (14)–(17), we can ultimately show that

$$\Omega^{(1)}(t) = -\frac{i\beta}{\epsilon_0 L^3}(Ze)\,\Omega^{(0)}\sum_{\mathbf{p}''}\frac{p''_\gamma}{p''^2}\int_0^{t} d\tau'''\, j_\gamma(\mathbf{p}'', t-\tau''')\exp[i\mathbf{p}''\cdot\mathbf{v}_0(t-\tau''')] \tag{18}$$

where

$$\mathbf{j}(\mathbf{p}'', t) = \int_{L^3} d^3\mathbf{r}\,\exp(-i\mathbf{p}''\cdot\mathbf{r})\,\mathbf{j}(\mathbf{r}, t) \tag{19}$$

is the $\mathbf{p}''$th Fourier component of the microscopic current density[3] and $\beta^{-1} = kT$ is the temperature in energy units. Equations (13d), (13e), and (18) therefore combine to yield the equilibrium triplet expressions:

$$U^{(3,1)}_{\mathrm{POL,STAT}}(t) = -\frac{i\beta(Ze)^3}{m_0\epsilon_0 L^3} v_{0\mu} \sum_{\mathbf{p}''} \frac{p''_\nu}{p''^2} \int_0^t d\tau \int_0^\tau d\tau' \int_0^{\tau'} d\tau'' \int_0^{\tau''} d\tau''' \times \{\exp[i\mathbf{p}''\cdot\mathbf{v}_0(t-\tau''')]\} \times \langle E_{\mu,\nu}(\mathbf{v}_0\tau,\tau)\, E_\nu(\mathbf{v}_0\tau'',\tau'')\, j_\nu(\mathbf{p}'', t-\tau'')\rangle^{(0)} \tag{20a}$$

$$U^{(3,2)}_{\mathrm{POL,STAT}}(t) = -\frac{i\beta(Ze)^3}{2m_0\epsilon_0 L^3} \sum_{\mathbf{p}''} \frac{p''_\nu}{p''^2} \int_0^t d\tau \int_0^t d\tau' \int_0^\infty d\tau''\, \exp[i\mathbf{p}''\cdot\mathbf{v}_0(t-\tau'')] \times \langle E_\mu(\mathbf{v}_0\tau,\tau)\, E_\mu(\mathbf{v}_0\tau',\tau')\, j_\nu(\mathbf{p}'', t-\tau'')\rangle^{(0)} \tag{20b}$$

The doublet and triplet correlations in (13b), (13c), (13f-g)–(13h), (20a), and (20b) comprise the set of equilibrium correlation tensors which will be treated in Section 4 for their conversion in Section 5 to linear and quadratic polarizability functions vis-à-vis appropriate fluctuation-dissipation theorems. We wish, however, first to cast the pure polarization energy loss contribution (13a) into dielectric form. This is taken up in the next section.

3. POLARIZATION CONTRIBUTIONS

Starting from Poisson's equation,

$$i\mathbf{p}\cdot\mathbf{D}(\mathbf{p},\mu) = (2\pi Ze/\epsilon_0)\,\delta(\mathbf{p}\cdot\mathbf{v}_0 - \mu) \tag{21}$$

for the electric induction response $\mathbf{D}(\mathbf{p},\mu)$ to the moving external charge, we present here a straightforward medium electrodynamic derivation of the dielectric formulation of the pure polarization energy loss contributions (13a) written in Fourier representation as

$$U^{(s,1)}_{\mathrm{POL}}(t) = -(Ze/2\pi L^3)\, v_{0\mu} \sum_{\mathbf{p}} \int_{-\infty}^{\infty} d\mu\, \langle E_\mu(\mathbf{p},\mu)\rangle^{(s)} \int_0^t d\tau\, \exp[i(\mathbf{p}\cdot\mathbf{v}_0 - \mu)\tau], \quad s = 1, 2 \tag{22}$$

[3] Throughout the remainder of this paper we adopt the spatial Fourier transform convention (19). The temporal transform is given by

$$\mathbf{j}(\mathbf{p}'',\mu'') = \int_{-\infty}^{\infty} dt \int_{L^3} d^3\mathbf{r}\, \exp[i(\mu''t - \mathbf{p}''\cdot\mathbf{r})]\,\mathbf{j}(\mathbf{r},t)$$

with inverse transform convention

$$\mathbf{j}(\mathbf{r},t) = (1/L^3)\sum_{\mathbf{p}''}\int_{-\infty}^{\infty} (d\mu''/2\pi)\, \exp[i(\mathbf{p}''\cdot\mathbf{r} - \mu''t)]\,\mathbf{j}(\mathbf{p}'',\mu'')$$

Now, the connection between $\mathbf{D}(\mathbf{p}, \mu)$ and the average electric field corrections $\langle \mathbf{E}(\mathbf{p}, \mu)\rangle^{(s)}$ is given for a homogeneous and stationary plasma by the constitutive relation[8]

$$\mathbf{D}(\mathbf{p}, \mu) = \boldsymbol{\varepsilon}(\mathbf{p}, \mu) \cdot \sum_{s=1,2} \langle \mathbf{E}(\mathbf{p}, \mu)\rangle^{(s)} + \frac{1}{L^3} \sum_{\mathbf{p}'} \int_{-\infty}^{\infty} \frac{d\mu'}{2\pi}\, \boldsymbol{\varepsilon}(\mathbf{p}', \mathbf{p}'', \mu', \mu'') : \langle \mathbf{E}(\mathbf{p}', \mu')\rangle^{(1)} \langle \mathbf{E}(\mathbf{p}'', \mu'')\rangle^{(1)}$$
$$\mathbf{p}' + \mathbf{p}'' = \mathbf{p}, \quad \mu' + \mu'' = \mu \tag{23}$$

so that elimination of $\mathbf{D}(\mathbf{p}, \mu)$ between (21) and (23) yields, to $O(Ze)$ and $O[(Ze)^2]$, respectively,[4]

$$\langle \mathbf{E}(\mathbf{p}, \mu)\rangle^{(1)} = -i\mathbf{p} \frac{2\pi Ze}{\epsilon_0 p^2} \frac{\delta(\mathbf{p} \cdot \mathbf{v}_0 - \mu)}{\epsilon_L(\mathbf{p}, \mu)} \tag{24a}$$

and

$$\langle \mathbf{E}(\mathbf{p}, \mu)\rangle^{(2)} = -\frac{1}{2\pi L^3}\, \mathbf{p} \sum_{\mathbf{p}'} \hat{p}_\beta' \hat{p}_\gamma'' \frac{\epsilon_Q(\mathbf{p}', \mathbf{p}'', \mu', \mu'')}{p\, \epsilon_L(\mathbf{p}, \mu)} \times \langle E_\beta(\mathbf{p}', \mu')\rangle^{(1)} \langle E_\gamma(\mathbf{p}'', \mu'')\rangle^{(1)} \tag{24b}$$

where

$$\epsilon_L(\mathbf{p}, \mu) = \hat{p}_\alpha \hat{p}_\beta \epsilon_{\alpha\beta}(\mathbf{p}, \mu) \tag{25a}$$

and

$$\epsilon_Q(\mathbf{p}', \mathbf{p}'', \mu', \mu'') = \hat{p}_\alpha \hat{p}_\beta' \hat{p}_\gamma'' \epsilon_{\alpha\beta\gamma}(\mathbf{p}', \mathbf{p}'', \mu', \mu'') \tag{25b}$$

are the longitudinal projections of the linear and quadratic dielectric tensors $\epsilon_{\alpha\beta}$ and $\epsilon_{\alpha\beta\gamma}$, and where we exploited the scalar character of the average field, namely

$$\langle \mathbf{E}(\mathbf{p}, \mu)\rangle^{(s)} = -i\mathbf{p}\langle \phi(\mathbf{p}, \mu)\rangle^{(s)}, \qquad s = 1, 2 \tag{26}$$

[4] We note that the external charge density is itself modified by the presence of the plasma particles so that

$$\rho_{\text{ext}}(\mathbf{x}, t) = Ze\, \delta(\mathbf{x} - \mathbf{v}_0 t - \Delta\mathbf{x})$$

Since $\langle \Delta\mathbf{x}\rangle$ is, at most, of order $(Ze)^2$, the average modification in the external charge density and, consequently, the corresponding average electric field reaction to this modification are of order $(Ze)^3$. Now (24b) shows that $\langle \mathbf{E}(\mathbf{p}, \mu)\rangle^{(2)} \sim O[(Ze)^2 \epsilon_Q]$ and in the Vlasov approximation, $\epsilon_Q \sim O[(\beta n e^2 r^2/\epsilon_0)e] \sim O(e)$ (see p. 86), so that $\langle \mathbf{E}(\mathbf{p}, \mu)\rangle^{(2)} \sim O[(Ze)^2 e]$. Since e and (Ze) are on the same smallness footing, a proper formulation of (24b) should take account of the modification in the external charge density. This has the effect of adding to (29b) terms of order $(Ze)^4$ containing factors like $1/[\epsilon_L(\mathbf{p}', \mathbf{p}' \cdot \mathbf{v}_0)\, \epsilon_L(\mathbf{p}'', \mathbf{p}'' \cdot \mathbf{v}_0)]$.

since the plasma is assumed to be magnetic field-free. Equations (24a) and (24b) then combine to give for the second-order average electric field correction

$$\langle \mathbf{E}(\mathbf{p}, \mu)\rangle^{(2)} = \frac{2\pi(Ze)^2}{L^3\epsilon_0^{\,2}}\, \mathbf{p}\delta(\mathbf{p}\cdot\mathbf{v}_0 - \mu) \sum_{\mathbf{p}'} \frac{\eta(\mathbf{p}', \mathbf{p}'', \mathbf{p}'\cdot\mathbf{v}_0, \mathbf{p}''\cdot\mathbf{v}_0)}{pp'p''} \tag{27}$$

where

$$\eta(\mathbf{p}', \mathbf{p}'', \mu', \mu'') = \frac{\epsilon_Q(\mathbf{p}', \mathbf{p}'', \mu', \mu'')}{\epsilon_l(\mathbf{p}, \mu)\,\epsilon_L(\mathbf{p}', \mu')\,\epsilon_L(\mathbf{p}'', \mu'')} \tag{28}$$

Thus upon inserting (24a) and (27) back into (22) and taking account of the odd parity of $\operatorname{Re}\eta(\mathbf{p}', \mathbf{p}''; \mathbf{p}'\cdot\mathbf{v}_0, \mathbf{p}''\cdot\mathbf{v}_0)$ under simultaneous sign reversal of its two wave vector arguments,[5] one obtains the following desired dielectric formulations of the energy loss corrections:

$$U^{(2)}_{\mathrm{POL}}(t) = -\frac{t(Ze)^2}{\epsilon_0 L^3} \sum_{\mathbf{p}} \frac{\mathbf{p}\cdot\mathbf{v}_0}{p^2} \operatorname{Im} \frac{1}{\epsilon_l(\mathbf{p}, \mathbf{p}\cdot\mathbf{v}_0)} \tag{29a}$$

and

$$U^{(3)}_{\mathrm{POL}}(t) = \frac{t(Ze)^3}{(\epsilon_0 L^3)^2} \sum_{\mathbf{p},\mathbf{p}'} \frac{\mathbf{p}\cdot\mathbf{v}_0}{pp'p''} \operatorname{Re}\eta(\mathbf{p}', \mathbf{p}'', \mathbf{p}'\cdot\mathbf{v}_0, \mathbf{p}''\cdot\mathbf{v}_0) \tag{29b}$$

$(\mathbf{p}' + \mathbf{p}'' = \mathbf{p})$. The corresponding power loss correction $P^{(n)}$, $n = 2, 3$, are then obtained from (29a) and (29b) and the definition (see footnote 2)

$$P^{(n)} = \lim_{t\to\infty} [U^{(n)}(t)/t] \tag{30}$$

The expression (29a) is the well-known result for the linear contribution to the polarization energy loss of a test particle. This result is corroborated by model-independent medium electrodynamics which predicts that dissipation in a lossy plasma is portrayed by the anti-Hermitian part of the

[5] Equations (25a), (25b), and (28), together with the reality of $\epsilon_{\alpha\beta}(\mathbf{r} - \mathbf{r}', t - t')$ and $\epsilon_{\alpha\beta\gamma}(\mathbf{r} - \mathbf{r}', \mathbf{r} - \mathbf{r}''; t - t', t - t'')$, suggest that

$$\eta(-\mathbf{p}', -\mathbf{p}''; -\mu', -\mu'') = \frac{\epsilon_Q(-\mathbf{p}', -\mathbf{p}''; -\mu', -\mu'')}{\epsilon_L(-\mathbf{p}, -\mu)\epsilon_l(-\mathbf{p}', -\mu')\epsilon_L(-\mathbf{p}'', -\mu'')}$$

$$= \frac{-\hat{p}_\alpha\hat{p}_\beta'\hat{p}_\gamma''\epsilon^*_{\alpha\beta\gamma}(\mathbf{p}', \mathbf{p}''; \mu', \mu'')}{\hat{p}_\mu\hat{p}_\eta\epsilon^*_{\mu\eta}(\mathbf{p}, \mu)\hat{p}_\nu'\hat{p}_\xi'\epsilon^*_{\nu\xi}(\mathbf{p}', \mu')\hat{p}_\lambda''\hat{p}_\theta''\epsilon^*_{\lambda\theta}(\mathbf{p}'', \mu'')}$$

$$= -\left\{\frac{\epsilon_Q(\mathbf{p}', \mathbf{p}''; \mu', \mu'')}{\epsilon_L(\mathbf{p}, \mu)\epsilon_l(\mathbf{p}', \mu')\epsilon_L(\mathbf{p}'', \mu'')}\right\}^* = -\eta^*(\mathbf{p}', \mathbf{p}''; \mu', \mu'')$$

or $\operatorname{Re}\eta(\mathbf{p}', \mathbf{p}''; \mu', \mu'') = -\operatorname{Re}\eta(-\mathbf{p}', -\mathbf{p}''; -\mu', -\mu'')$ and $\operatorname{Im}\eta(\mathbf{p}', \mathbf{p}''; \mu', \mu'') = \operatorname{Im}\eta(-\mathbf{p}', -\mathbf{p}''; -\mu', -\mu'')$.

dielectric tensor.[11] Turning now to (29b), it appears that only Re η contributes to the quadratic polarization correction. Later on we shall see, however, that the statistical energy loss contribution involves both the real and imaginary parts. There are no known energy loss calculations made from nonlinear medium electrodynamics which can support these findings. Their consequences with respect to lossless plasmas will be discussed in Section 6. We consider next the statistical energy loss contribution.

4. STATISTICAL CONTRIBUTION

In order to replace ultimately the equilibrium microscopic correlations in (13b), (13c), (13f)-(13h), (20a), (20b) by their dielectric function relatives supplied by appropriate fluctuation-dissipation theorems, it is necessary first to formulate these correlations purely in terms of microscopic current densities. This is the task of the present section and is, of course, facilitated by converting space time correlations to correlations of wave vector and frequency components. Thus conversion to Fourier components via the inverse transform convention of footnote 3 and use of Poisson's equation for the microscopic electric field, e.g.,

$$\mathbf{E}(\mathbf{p}, \mu) = (i/\mu\epsilon_0)\,\hat{p}\hat{p}\cdot\mathbf{j}(\mathbf{p}, \mu), \qquad \hat{p} = \mathbf{p}/p \tag{31}$$

leads to the following expressions for (13b), (13c), (13f)-(13h), (20a), and (20b):

$$U^{(2,m)}(t) = \frac{(Ze)^2}{m_0(2\pi L^3\epsilon_0)^2}\sum_{\mathbf{p}}\hat{p}_\alpha\hat{p}_\beta\int_{-\infty}^{\infty}\frac{d\mu}{\mu}\int_{-\infty}^{\infty}\frac{d\mu'}{\mu}\langle j_\alpha(\mathbf{p}, \mu)\,j_\beta(-\mathbf{p}, \mu')\rangle^{(0)}$$
$$\times\, I_{2,m}(\mathbf{p}, \mu; t), \qquad m = 1, 2 \tag{32a}$$

$$U^{(3,n)}(t) = \frac{(Ze)^3}{m_0^2(2\pi L^3\epsilon_0)^3}\sum_{\mathbf{p},\mathbf{p}',\mathbf{p}''}\frac{(\hat{p}\cdot\hat{p}')}{p''}\hat{p}_\alpha\hat{p}_\beta'\hat{p}_\gamma''\int_{-\infty}^{\infty}\frac{d\mu}{\mu}\int_{-\infty}^{\infty}\frac{d\mu'}{\mu'}\int_{-\infty}^{\infty}d\mu''$$
$$\times\,\langle j_\alpha(\mathbf{p}, \mu)\,j_\beta(\mathbf{p}', \mu')\,j_\gamma(\mathbf{p}'', \mu'')\rangle^{(0)}\, I_{3,n}(\mathbf{p}, \mathbf{p}', \mathbf{p}'', \mu, \mu', \mu''; t) \tag{32b}$$

($n = 1, 2,\ldots, 5$), where

$$I_{2,1}(\mathbf{p}, \mu; t) = i\mathbf{p}\cdot\mathbf{v}_0\int_0^t d\tau\,\exp[i(\mathbf{p}\cdot\mathbf{v}_0 - \mu)\tau]$$
$$\times\int_0^{\tau} d\tau'\int_0^{\tau'} d\tau''\,\exp[-i(\mathbf{p}\cdot\mathbf{v}_0 - \mu)\tau''] \tag{33a}$$

$$I_{2,2}(\mathbf{p}, \mu; t) = \tfrac{1}{2}\int_0^t d\tau\int_0^t d\tau'\,\exp[i(\mathbf{p}\cdot\mathbf{v}_0 - \mu)(\tau - \tau')] \tag{33b}$$

$$
\begin{aligned}
I_{3,1}(\mathbf{p}, \mathbf{p}', \mathbf{p}'', \mu, \mu', \mu''; t) = {} & -\beta m_0(\mathbf{p} \cdot \mathbf{v}_0) \int_0^{\infty} d\tau''' \exp[i(\mathbf{p}'' \cdot \mathbf{v}_0 - \mu'')(t - \tau''')] \\
& \times \int_0^{t} d\tau \exp[i(\mathbf{p} \cdot \mathbf{v}_0 - \mu)\tau] \\
& \times \int_0^{\tau} d\tau' \int_0^{\tau'} d\tau'' \exp[i(\mathbf{p}' \cdot \mathbf{v}_0 - \mu')\tau''] \qquad (33c)
\end{aligned}
$$

$$
\begin{aligned}
I_{3,2}(\mathbf{p}, \mathbf{p}', \mathbf{p}'', \mu, \mu', \mu''; t) = {} & \tfrac{1}{2} i\beta m_0 \int_0^{\infty} d\tau'' \exp[i(\mathbf{p}'' \cdot \mathbf{v}_0 - \mu'')(t - \tau'')] \\
& \times \int_0^{t} d\tau \exp[i(\mathbf{p} \cdot \mathbf{v}_0 - \mu)\tau] \\
& \times \int_0^{t} d\tau' \exp[i(\mathbf{p}' \cdot \mathbf{v}_0 - \mu')\tau'] \qquad (33d)
\end{aligned}
$$

$$
\begin{aligned}
I_{3,3}(\mathbf{p}, \mathbf{p}', \mathbf{p}'', \mu, \mu', \mu''; t) = {} & (\mathbf{p}' \cdot \mathbf{p}''/\mu'') \int_0^{t} d\tau \exp[i(\mathbf{p} \cdot \mathbf{v}_0 - \mu)\tau] \\
& \times \int_0^{t} d\tau' \exp[i(\mathbf{p}' \cdot \mathbf{v}_0 - \mu')\tau'] \\
& \times \int_0^{\tau'} d\tau'' \int_0^{\tau''} d\tau''' \exp[i(\mathbf{p}'' \cdot \mathbf{v}_0 - \mu'')\tau'''] \qquad (33e)
\end{aligned}
$$

$$
\begin{aligned}
I_{3,4}(\mathbf{p}, \mathbf{p}', \mathbf{p}'', \mu, \mu', \mu''; t) = {} & [i(\mathbf{p} \cdot \mathbf{v}_0)(\mathbf{p}' \cdot \mathbf{p}'')/\mu''] \int_0^{t} d\tau \exp[i(\mathbf{p} \cdot \mathbf{v}_0 - \mu)\tau] \\
& \times \int_0^{\tau} d\tau' \int_0^{\tau'} d\tau'' \exp[i(\mathbf{p}' \cdot \mathbf{v}_0 - \mu')\tau''] \\
& \times \int_0^{\tau''} d\tau''' \int_0^{\tau'''} d\tau^{\mathrm{IV}} \exp[i(\mathbf{p}'' \cdot \mathbf{v}_0 - \mu'')\tau^{\mathrm{IV}}] \qquad (33f)
\end{aligned}
$$

$$
\begin{aligned}
I_{3,5}(\mathbf{p}, \mathbf{p}', \mathbf{p}'', \mu, \mu', \mu''; t) = {} & [i(\mathbf{p} \cdot \mathbf{v}_0)(\mathbf{p} \cdot \mathbf{p}'')/2\mu''] \int_0^{t} d\tau \exp[i(\mathbf{p} \cdot \mathbf{v}_0 - \mu)\tau] \\
& \times \int_0^{\tau} d\tau' \int_0^{\tau'} d\tau'' \exp[i(\mathbf{p}' \cdot \mathbf{v}_0 - \mu')\tau''] \\
& \times \int_0^{\tau} d\tau''' \int_0^{\tau'''} d\tau^{\mathrm{IV}} \exp[i(\mathbf{p}'' \cdot \mathbf{v}_0 - \mu'')\tau^{\mathrm{IV}}] \qquad (33g)
\end{aligned}
$$

In the long-time ($t \to \infty$) limit, one can show that the I integrals (33a)–(33c), (33f), and (33g) become

$$\lim_{t \to \infty} I_{2,1}(\mathbf{p}, \mu; t) = 2\pi t(\hat{p} \cdot \mathbf{v}_0)\left(\hat{p} \cdot \frac{\partial}{\partial \mathbf{v}_0}\right) \delta_+(\mathbf{p} \cdot \mathbf{v}_0 - \mu) \qquad (34a)$$

$$\lim_{t \to \infty} I_{2,2}(\mathbf{p}, \mu; t) = \pi t\, \delta(\mathbf{p} \cdot \mathbf{v}_0 - \mu) \qquad (34b)$$

$$
\begin{aligned}
\lim_{t \to \infty} I_{3,1}(\mathbf{p}', \mathbf{p}'', \mu', \mu''; t) = {} & -2\pi\beta m_0 t(\mathbf{p} \cdot \mathbf{v}_0) \frac{\partial}{\partial v_{0\lambda}} \\
& \times \left[\frac{T_{\lambda m}(\hat{p}'')\,\hat{p}_m'}{p'\hat{p}_r'\hat{p}_s' T_{rs}(\hat{p}'')} \frac{\delta(\mathbf{p}'' \cdot \mathbf{v}_0 - \mu'')}{\mathbf{p}' \cdot \mathbf{v}_0 - \mu'}\right] \qquad (34c)
\end{aligned}
$$

$$\lim_{t\to\infty} I_{3,4}(\mathbf{p}', \mathbf{p}'', \mu', \mu''; t) = \frac{(2\pi)^2 it(\mathbf{p}\cdot\mathbf{v}_0)(\mathbf{p}'\cdot\mathbf{p}'')}{\mu'' p p''}$$

$$\times \frac{\partial^2}{\partial v_{0a}\,\partial v_{0b}} \left[\frac{T_{al}(\hat{p})\hat{p}''_l T_{bm}(\hat{p}'')\hat{p}_m}{\hat{p}_r \hat{p}_s T_{rs}(\hat{p}'')\hat{p}''_t \hat{p}''_u T_{tu}(\hat{p})} \right.$$

$$\left. \times\, \delta_+(\mathbf{p}\cdot\mathbf{v}_0 - \mu)\,\delta_-(\mathbf{p}''\cdot\mathbf{v}_0 - \mu'') \right] \qquad (34d)$$

$$\lim I_{3,5}(\mathbf{p}', \mathbf{p}'', \mu', \mu''; t) = \frac{(2\pi i)^2 it(\mathbf{p}\cdot\mathbf{v}_0)(\mathbf{p}\cdot\mathbf{p}'')}{2\mu'' p' p''} \frac{\partial^2}{\partial v_{0a}\,\partial v_{0b}}$$

$$\times \left[\frac{T_{al}(\hat{p}')\hat{p}''_l T_{bm}(\hat{p}'')\hat{p}_m'}{\hat{p}_r'\hat{p}_s T_{rs}(\hat{p}'')\hat{p}_t\hat{p}_u T_{tu}(\hat{p}')} \right.$$

$$\left. \times\, \delta_+(\mathbf{p}'\cdot\mathbf{v}_0 - \mu')\,\delta_+(\mathbf{p}''\cdot\mathbf{v}_0 - \mu'') \right] \qquad (34e)$$

where, e.g., $T_{\lambda m}(\hat{p}'') = \delta_{\lambda m} - \hat{p}''_\lambda \hat{p}''_m$ is the transverse projection tensor with respect to the unit wave vector $\hat{p}''$, and

$$\delta_\pm(\mathbf{p}\cdot\mathbf{v}_0 - \mu) = \tfrac{1}{2}\delta(\mathbf{p}\cdot\mathbf{v}_0 - \mu) \pm (i/2\pi)\,P(1/\mathbf{p}\cdot\mathbf{v}_0 - \mu)$$

P denoting the Cauchy principal part. In the $t \to \infty$ limit, the integral of (33d) and that of $I_{3,3}$ which contributes to $U^{(3,3)}$ can be shown to be time-independent. Consequently $U^{(3,2)}$ and $U^{(3,3)}$, unlike the other energy loss contributions, remain bounded as t tends to infinity and are therefore neglected in the sequel.

4.1. Doublet Correlations

Let us consider first the statistical contributions (32a) to the test charge energy loss arising from the equilibrium doublet current correlations. Equation (32) with $m = 1$ and Eq. (34a) combine to give

$$\lim U^{(2,1)}_{\mathrm{STAT}}(t) = \frac{2\pi t(Ze)^2}{m_0(2\pi\epsilon_0 L^3)^2} \sum_{\mathbf{p}} \frac{1}{p^2}(\mathbf{p}\cdot\mathbf{v}_0)\left(\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{v}_0}\right)\hat{p}_\alpha\hat{p}_\beta$$

$$\times \int_{-\infty}^{\infty} \frac{d\mu}{\mu} \int_{-\infty}^{\infty} \frac{d\mu'}{\mu'} \langle j_\alpha(\mathbf{p},\mu)\,j_\beta(-\mathbf{p},\mu')\rangle^{(0)}\,\delta_+(\mathbf{p}\cdot\mathbf{v}_0 - \mu) \qquad (35)$$

We observe here that $\langle j_\alpha(\mathbf{p},\mu)\,j_\beta(-\mathbf{p},\mu')\rangle^{(0)}$ is even and real; its sign remains unchanged under separate wave vector and frequency reversals, since the equilibrium system is reflection-invariant and the sign of the equilibrium current-current correlations is unchanged under microscopic time reversal. Thus the operator

$$f(\mathbf{p},\mu,\mu') = \frac{(\mathbf{p}\cdot\mathbf{v}_0)\,\hat{p}_\alpha\hat{p}_\beta}{\mu\mu'} \langle j_\alpha(\mathbf{p},\mu)\,j_\beta(-\mathbf{p},\mu')\rangle^{(0)}\,\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{v}_0} \qquad (36)$$

has net even parity with respect to simultaneous reversal of $\mathbf{p}$, μ, and μ'. Clearly, then, the product of $f(\mathbf{p}, \mu, \mu')$ and the principal value part of $\delta_+(\mathbf{p}\cdot\mathbf{v}_0 - \mu)$ has net odd parity in the sense just described, so that this product contributes nothing under the simultaneous summation and integration operations of (35). Only the product of f and the δ-function part can contribute, so that (35) simplifies to

$$\lim_{t\to\infty} U^{(2,1)}_{\text{STAT}}(t) = \frac{\pi t(Ze)^2}{m_0(2\pi\epsilon_0 L^3)^2} \sum_{\mathbf{p}} \frac{1}{p^2} (\mathbf{p}\cdot\mathbf{v}_0) \left(\mathbf{p}\cdot\frac{\partial}{\partial \mathbf{v}_0}\right) \frac{\hat{p}_\alpha \hat{p}_\beta}{(\mathbf{p}\cdot\mathbf{v}_0)^2} \times \int_{-\infty}^{\infty} d\mu' \, \langle j_\alpha(\mathbf{p}, \mathbf{p}\cdot\mathbf{v}_0) j_\beta(-\mathbf{p}, \mu')\rangle^{(0)} \tag{37}$$

Concerning $U^{(2,2)}_{\text{STAT}}$, Eq. (32a) with $m = 2$ and Eq. (34b) combine to readily yield

$$\lim_{t\to\infty} U^{(2,2)}_{\text{STAT}}(t) = \frac{-\pi t(Ze)^2}{m_0(2\pi\epsilon_0 L^3)^2} \sum_{\mathbf{p}} \frac{\hat{p}_\alpha \hat{p}_\beta}{(\mathbf{p}\cdot\mathbf{v}_0)^2} \int_{-\infty}^{\infty} d\mu' \, \langle j_\alpha(\mathbf{p}, \mathbf{p}\cdot\mathbf{v}_0) j_\beta(-\mathbf{p}, \mu')\rangle^{(0)} \tag{38}$$

Then upon adding (37) and (38), one recovers Kalman's *linear* results for the statistical contributions to the energy loss.

4.2. Triplet Correlations

We turn next to the reduction of the triplet contributions (32b) to simplified forms similar to (37) and (38). Consider first the mixed polarization-statistical energy loss term, (32b) with $n = 1$. Upon combining this with (34c) and letting $\mathbf{p}'' \to \mathbf{p}$, $\mathbf{p}' \to \mathbf{p}''$, $\mu' \to \mu''$, $\mu \to \mu'$ for future notational convenience, we obtain

$$\begin{aligned}\lim_{t\to\infty} & U^{(3,1)}_{\text{POL,STAT}}(t) \\ &= \frac{2\pi t\beta(Ze)^3}{m_0(2\pi\epsilon_0 L^3)^3} \sum_{\mathbf{p},\mathbf{p}''} \frac{[(\mathbf{p}+\mathbf{p}'')\cdot\mathbf{p}''][(\mathbf{p}+\mathbf{p}'')\cdot\mathbf{v}_0](\mathbf{p}\cdot\mathbf{v}_0)\, p_\alpha(\mathbf{p}+\mathbf{p}'')_\beta\, p''_\gamma}{(pp'')^3\, |\mathbf{p}+\mathbf{p}''|^2} \\ &\times \frac{T_{\lambda l}(\hat{p})\,\hat{p}''_l\, T_{bm}(\hat{p}'')\, p_m}{\hat{p}_r \hat{p}_s T_{rs}(\hat{p}'')\,\hat{p}''_t \hat{p}''_u T_{tu}(\hat{p})}\, p_b \frac{\partial}{\partial v_{0\lambda}} \left[\frac{1}{\mathbf{p}\cdot\mathbf{v}_0} - \int_{-\infty}^{\infty} \frac{d\mu'}{\mu'} \int_{\infty}^{\infty} \frac{d\mu''}{\mu''} \right. \\ &\left. \times \langle j_\alpha(\mathbf{p}, \mathbf{p}\cdot\mathbf{v}_0) j_\beta(-\mathbf{p}-\mathbf{p}'', \mu') j_\gamma(\mathbf{p}'', \mu'')\rangle^{(0)} P \frac{1}{\mathbf{p}''\cdot\mathbf{v}_0 - \mu''}\right]\end{aligned} \tag{39}$$

To see why $1/(\mathbf{p}'\cdot\mathbf{v}_0 - \mu')$ in (34c) was interpreted as being a Cauchy principal part, we note first that the product

$$[(\mathbf{p}\cdot\mathbf{p}')\,\hat{p}_\alpha \hat{p}_\beta' \hat{p}_\gamma''/pp'p''\mu\mu']\langle j_\alpha(\mathbf{p}, \mu) j_\beta(\mathbf{p}', \mu') j_\gamma(\mathbf{p}'', \mu'')\rangle^{(0)} \tag{40a}$$

appearing in (32b) with $n = 1$ undergoes a change in sign under simultaneous sign reversal of the wave vector and frequency variables. Consequently, only the product of (40a) with the corresponding odd parity part of $I_{3,1}$ can contribute to (32b) with $n = 1$. Now let us assume that the interpretation of $1/(\mathbf{p}' \cdot \mathbf{v}_0 - \mu')$ is entirely arbitrary and suppose therefore that a vanishingly small quantity $i\xi$ can be added to or subtracted from $\mathbf{p}' \cdot \mathbf{v}_0 - \mu'$. Then

$$\lim_{\xi \to 0} \frac{1}{\mathbf{p}' \cdot \mathbf{v}_0 - \mu' \pm i\xi} = \mp \pi i\, \delta(\mathbf{p}' \cdot \mathbf{v}_0 - \mu') + P \frac{1}{\mathbf{p}' \cdot \mathbf{v}_0 - \mu'} \tag{40b}$$

and upon combining (34c) and (40b), it is clear that the ensuing δ-function part has net even parity with respect to the sign reversal of $(\mathbf{p}', \mathbf{p}'', \mu', \mu'')$, whereas the ensuing principal part contribution has odd parity. Thus only this latter can contribute to the energy loss expression (32b) with $n = 1$.

It now remains to evaluate the pure statistical contributions, (32b) with $n = 4, 5$. In the case of $U_{\mathrm{STAT}}^{(3,4)}$, the combination of (32b) with $n = 4$ and (34d) can be shown to yield in the limit $t \to \infty$

$$\begin{aligned}
\lim_{t \to \infty} U_{\mathrm{STAT}}^{(3,4)}(t) = {} & \frac{\pi t (Ze)^3}{m_0^2 (2\pi\epsilon_0 L^3)^3} \sum_{\mathbf{p},\mathbf{p}''} \frac{[(\mathbf{p} - \mathbf{p}'') \cdot \mathbf{v}_0][(\mathbf{p} + \mathbf{p}'') \cdot \mathbf{p}][(\mathbf{p} + \mathbf{p}'') \cdot \mathbf{p}'']}{(pp'')^3 \, |\mathbf{p} + \mathbf{p}''|^2} \\
& \times p_\alpha (\mathbf{p} + \mathbf{p}'')_\beta\, p''_\gamma \frac{T_{al}(\hat{p}) \hat{p}''_l T_{bm}(\hat{p}'') \hat{p}_m}{\hat{p}_r \hat{p}_s T_{rs}(\hat{p}'')\, \hat{p}''_t \hat{p}''_u T_{tu}(\hat{p})} \\
& \times \frac{\partial^2}{\partial v_{0a}\, \partial v_{0b}} \left[\frac{1}{\mathbf{p} \cdot \mathbf{v}_0} \int_{-\infty}^{\infty} \frac{d\mu'}{\mu'} \int_{-\infty}^{\infty} \frac{d\mu''}{\mu''} \right. \\
& \left. \times \langle j_\alpha(\mathbf{p}, \mathbf{p} \cdot \mathbf{v}_0)\, j_\beta(-\mathbf{p} - \mathbf{p}'', \mu')\, j_\gamma(\mathbf{p}'', \mu'') \rangle^{(0)} P \frac{1}{\mathbf{p}'' \cdot \mathbf{v}_0 - \mu''} \right]
\end{aligned} \tag{41}$$

Similarly, for $U_{\mathrm{STAT}}^{(3,5)}$, the combination of (32b) with $n = 5$ and (34e), accompanied by the summation-integration variable transformations

$$\mathbf{p}' \to \mathbf{p}'', \qquad \mathbf{p}'' \to \mathbf{p}, \qquad \mu' \to \mu'', \qquad \mu \to \mu'$$

ultimately results in the expression

$$\begin{aligned}
\lim_{t \to \infty} U_{\mathrm{STAT}}^{(3,5)}(t) = {} & \frac{-\pi t (Ze)^3}{m_0^2 (2\pi\epsilon_0 L^3)^3} \sum_{\mathbf{p},\mathbf{p}''} \frac{[(\mathbf{p} + \mathbf{p}'') \cdot \mathbf{v}_0][(\mathbf{p} + \mathbf{p}'') \cdot \mathbf{p}][(\mathbf{p} + \mathbf{p}'') \cdot \mathbf{p}'']}{(pp'')^3 \, |\mathbf{p} + \mathbf{p}''|^2} \\
& \times p_\alpha (\mathbf{p} + \mathbf{p}'')_\beta\, p''_\gamma \frac{T_{al}(\hat{p}) \hat{p}''_l T_{bm}(\hat{p}'') \hat{p}_m}{\hat{p}_r \hat{p}_s T_{rs}(\hat{p}'')\, \hat{p}''_t \hat{p}''_u T_{tu}(\hat{p})} \\
& \times \frac{\partial^2}{\partial v_{0a}\, \partial v_{0b}} \left[\frac{1}{\mathbf{p} \cdot \mathbf{v}_0} \int_{-\infty}^{\infty} \frac{d\mu'}{\mu'} \int_{-\infty}^{\infty} \frac{d\mu''}{\mu''} \right. \\
& \left. \times \langle j_\alpha(\mathbf{p}, \mathbf{p} \cdot \mathbf{v}_0)\, j_\beta(-\mathbf{p} - \mathbf{p}'', \mu')\, j_\gamma(\mathbf{p}'', \mu'') \rangle^{(0)} P \frac{1}{\mathbf{p}'' \cdot \mathbf{v}_0 - \mu''} \right]
\end{aligned} \tag{42}$$

which is strikingly similar to (41).

To summarize this section, we add (37) and (38) to obtain the following statistical contribution to the power loss [see Eq. (30)] due to the doublet current density correlations:

$$P_{\mathrm{STAT}}^{(2)} = \frac{-\pi(Ze)^2}{m_0(2\pi\epsilon_0 L^3)^2} \sum_{\mathbf{p}} \left(1 + \frac{\mathbf{p}\cdot\mathbf{v}_0}{p^2}\,\mathbf{p}\cdot\frac{\partial}{\partial \mathbf{v}_0}\right) \frac{\hat{p}_\alpha \hat{p}_\beta}{(\mathbf{p}\cdot\mathbf{v}_0)^2} \times \int_{-\infty}^{\infty} d\mu \langle j_\alpha(\mathbf{p}, \mathbf{p}\cdot\mathbf{v}_0)\, j_\beta(-\mathbf{p}, \mu)\rangle^{(0)} \tag{43}$$

and we add (39), (41), and (42) to obtain the following mixed and pure statistical contributions due to the triplet current correlations:

$$\begin{aligned} P_{\mathrm{POL\text{-}STAT+STAT}}^{(3)} = {} & \frac{2\pi(Ze)^3}{m_0^2(2\pi\epsilon_0 L^3)^3} \sum_{\mathbf{p}',\mathbf{p}''} \frac{[(\mathbf{p}' + \mathbf{p}'')\cdot\mathbf{p}'']\, p_\alpha'(\mathbf{p}' + \mathbf{p}'')_\beta\, p_\gamma''}{(p'p'')^3\,|\mathbf{p}' + \mathbf{p}''|^2} \\ & \times \frac{T_{al}(\hat{p}')\,\hat{p}_l''\,T_{bm}(\hat{p}'')\,\hat{p}_m'}{\hat{p}_r'\hat{p}_s'\,T_{rs}(\hat{p}'')\,\hat{p}_t''\hat{p}_u''\,T_{tu}(\hat{p}')} \Big\{\beta m_0(\mathbf{p}'\cdot\mathbf{v}_0)[(\mathbf{p}' + \mathbf{p}'')\cdot\mathbf{v}_0]\, p_b'\frac{\partial}{\partial v_{0a}} \\ & - [(\mathbf{p}' + \mathbf{p}'')\cdot\mathbf{p}'](\mathbf{p}''\cdot\mathbf{v}_0)\frac{\partial^2}{\partial v_{0a}\,\partial v_{0a}}\Big\}\Big\{\frac{1}{\mathbf{p}'\cdot\mathbf{v}_0}\int_{-\infty}^{\infty}\frac{d\mu'}{\mu'}\int_{-\infty}^{\infty}\frac{d\mu''}{\mu''} \\ & \times P\frac{1}{\mathbf{p}''\cdot\mathbf{v}_0 - \mu''}\langle j_\alpha(\mathbf{p}', \mathbf{p}'\cdot\mathbf{v}_0)\, j_\beta(-\mathbf{p}' - \mathbf{p}'', \mu')\, j_\gamma(\mathbf{p}'', \mu'')\rangle^{(0)}\Big\} \end{aligned} \tag{44}$$

We have thus arrived at the point where the microscopic current density fluctuation spectra in (43) and (44) are to be replaced by appropriate polarizability functions via fluctuation-dissipation theorems. This is taken up in the next section and our calculations should ultimately lead to energy loss expressions similar to the pure polarization results (29a) and (29b).

5. FLUCTUATION-DISSIPATION THEOREMS AND DIELECTRIC FORMULATION OF THE FORM FACTORS

It is notationaly convenient to introduce here the scalar "form factors"

$$S^{(2)}(\mathbf{p}) = 2\pi L^3 Q^{(2)}(\mathbf{p}, \mathbf{p}\cdot\mathbf{v}_0) \tag{45}$$

$$S^{(3)}(\mathbf{p}', \mathbf{p}'') = -\int_{-\infty}^{\infty} \frac{d\mu''}{\mu''}\, Q^{(3)}(\mathbf{p}', \mathbf{p}''; \mathbf{p}'\cdot\mathbf{v}_0, \mu'')\frac{1}{\mathbf{p}'\cdot\mathbf{v}_0 + \mu''}\, P\frac{1}{\mathbf{p}'\cdot\mathbf{v}_0 - \mu''} \tag{46}$$

where

$$2\pi L^3 Q^{(2)}(\mathbf{p}, \mathbf{p}\cdot\mathbf{v}_0)\,\delta(\mathbf{p}\cdot\mathbf{v}_0 + \mu) = \hat{p}_\alpha\hat{p}_\beta\langle j_\alpha(\mathbf{p}, \mathbf{p}\cdot\mathbf{v}_0)\, j_\beta(-\mathbf{p}, \mu)\rangle^{(0)} \tag{47}$$

$$\begin{aligned} & Q^{(3)}(\mathbf{p}', \mathbf{p}''; \mathbf{p}'\cdot\mathbf{v}_0, \mu'')\,\delta(\mu' + \mathbf{p}'\cdot\mathbf{v}_0 + \mu'') \\ & = \frac{p_\alpha'(\mathbf{p}' + \mathbf{p}'')_\beta\, p_\gamma''}{p'\,|\mathbf{p}' + \mathbf{p}''|\,p''}\langle j_\alpha(\mathbf{p}', \mathbf{p}'\cdot\mathbf{v}_0)\, j_\beta(-\mathbf{p}' - \mathbf{p}'', \mu')\, j_\gamma(\mathbf{p}'', \mu'')\rangle^{(0)} \end{aligned} \tag{48}$$

and it is clear that (45) and (46) written also as

$$S^{(3)}(\mathbf{p}', \mathbf{p}'') = \frac{p_\alpha'(\mathbf{p}' + \mathbf{p}'')_\beta p_\gamma''}{p' |\mathbf{p}' + \mathbf{p}''| p''} \int_{-\infty}^{\infty} \frac{d\mu'}{\mu'} \int_{-\infty}^{\infty} \frac{d\mu''}{\mu''} P \frac{1}{\mathbf{p}'' \cdot \mathbf{v}_0 - \mu''} \times \langle j_\alpha(\mathbf{p}', \mathbf{p}' \cdot \mathbf{v}_0) j_\beta(-\mathbf{p}' - \mathbf{p}'', \mu') j_\gamma(\mathbf{p}'', \mu'') \rangle^{(0)} \tag{49}$$

are the central terms of (43) and (44). In this section we shall recast (45) and (46) into dielectric form.

The starting points for our analysis are the following linear and quadratic fluctuation-dissipation theorems of plasma physics[7,9]:

$$Q^{(2)}(\mathbf{p}, \mu) = (2/\beta) \operatorname{Re} \hat{\sigma}_L(\mathbf{p}, \mu) \tag{50a}$$

and

$$Q^{(3)}(\mathbf{p}', \mathbf{p}''; \mu', \mu'') = -(8\pi L^3/\beta^2) \operatorname{Re}[\hat{\sigma}_Q(\mathbf{p}', \mathbf{p}''; \mu', \mu'') - \hat{\sigma}_Q(\mathbf{p}', -\mathbf{p}; -\mu', \mu) - \hat{\sigma}_Q(-\mathbf{p}, \mathbf{p}''; \mu, -\mu'')] \tag{50b}$$

($\mathbf{p} = \mathbf{p}' + \mathbf{p}''$, $\mu = \mu' + \mu''$), where $\hat{\sigma}_L$ and $\hat{\sigma}_Q$ are the so-called linear and quadratic *external* conductivities which connect the average first- and second-order induced current density responses, respectively, to the external driving field $\hat{\mathbf{E}}$ [of the moving test charge see Eq. (17)] and the product $\hat{\mathbf{E}}\hat{\mathbf{E}}$, namely

$$\langle j_\mu(\mathbf{p}, \mu) \rangle^{(1)} = \hat{\sigma}_{\mu\nu}(\mathbf{p}, \mu) \hat{E}_\nu(\mathbf{p}, \mu) \tag{51a}$$

$$\langle j_\mu(\mathbf{p}, \mu) \rangle^{(2)} = (1/L^3) \sum_{\mathbf{p}'} \int_{-\infty}^{\infty} \frac{d\mu'}{2\pi} \hat{\sigma}_{\mu\nu\lambda}(\mathbf{p}', \mathbf{p}''; \mu', \mu'') \hat{E}_\nu(\mathbf{p}', \mu') \hat{E}_\lambda(\mathbf{p}'', \mu'') \tag{51b}$$

($\mathbf{p} = \mathbf{p}' + \mathbf{p}''$, $\mu = \mu' + \mu''$), with

$$\hat{\sigma}_L(\mathbf{p}, \mu) = \hat{p}_\mu \hat{p}_\nu \hat{\sigma}_{\mu\nu}(\mathbf{p}, \mu) \tag{52a}$$

and

$$\hat{\sigma}_Q(\mathbf{p}', \mathbf{p}''; \mu', \mu'') = \hat{p}_\mu \hat{p}_\nu' \hat{p}_\lambda'' \hat{\sigma}_{\mu\nu\lambda}(\mathbf{p}', \mathbf{p}''; \mu', \mu'') \tag{52b}$$

Recent medium electrodynamic studies[7d,8] show that these external conductivities are related to the dielectric function (25a) and (25b) as follows:

$$\hat{\sigma}_L(\mathbf{p}, \mu) = -i\epsilon_0 \mu \frac{\alpha_L(\mathbf{p}, \mu)}{\epsilon_L(\mathbf{p}, \mu)} \tag{53a}$$

and

$$\hat{\sigma}_Q(\mathbf{p}', \mathbf{p}''; \mu', \mu'') = -i\epsilon_0 \mu \eta(\mathbf{p}', \mathbf{p}''; \mu', \mu''), \qquad \mu = \mu' + \mu'' \tag{53b}$$

where $\alpha_L(\mathbf{p}, \mu) = \epsilon_L(\mathbf{p}, \mu) - 1$ is the linear polarizability of the plasma medium and η is defined earlier through (28). Equations (50a) and (50b) permit us to write (46) and (47) as

$$S^{(2)}(\mathbf{p}) = (4\pi L^3/\beta) \operatorname{Re} \hat{\sigma}_L(\mathbf{p}, \mathbf{p} \cdot \mathbf{v}_0) \tag{54}$$

$$\begin{aligned} S^{(3)}(\mathbf{p}', \mathbf{p}'') = \frac{8\pi L^3}{\beta^2} \int_{-\infty}^{\infty} & \frac{d\mu''}{\mu''(\mathbf{p}' \cdot \mathbf{v}_0 + \mu'')} P \frac{1}{\mathbf{p}'' \cdot \mathbf{v}_0 - \mu''} \\ & \times \operatorname{Re}[\hat{\sigma}_Q(\mathbf{p}', \mathbf{p}''; \mathbf{p}' \cdot \mathbf{v}_0, \mu'') \\ & - \hat{\sigma}_Q(\mathbf{p}', -\mathbf{p}' - \mathbf{p}''; -\mathbf{p}' \cdot \mathbf{v}_0, \mu'' + \mathbf{p}' \cdot \mathbf{v}_0) \\ & - \hat{\sigma}_Q(-\mathbf{p}' - \mathbf{p}'', \mathbf{p}''; \mu'' + \mathbf{p}' \cdot \mathbf{v}_0, -\mu'')] \end{aligned} \tag{55}$$

Now let us examine more closely the nature of the denominator term, $1/[\mu''(\mathbf{p}' \cdot \mathbf{v}_0 + \mu'')]$, appearing in (55). This term might be interpreted to be any one of the four products,

$$[-2\pi i\, \delta_+(\mu'')][-2\pi i\, \delta_+(\mathbf{p}' \cdot \mathbf{v}_0 + \mu'')] \tag{56a}$$

$$[-2\pi i\, \delta_+(\mu'')][2\pi i\, \delta_-(\mathbf{p}' \cdot \mathbf{v}_0 + \mu'')] \tag{56b}$$

$$[2\pi i\, \delta_-(\mu'')][-2\pi i\, \delta_+(\mathbf{p}' \cdot \mathbf{v}_0 + \mu'')] \tag{56c}$$

$$[2\pi i\, \delta_-(\mu'')][2\pi i\, \delta_-(\mathbf{p}' \cdot \mathbf{v}_0 + \mu'')] \tag{56d}$$

Consider, e.g., (56c), which expands to

$$\begin{aligned} & \pi^2\, \delta(\mu'')\, \delta(\mathbf{p}' \cdot \mathbf{v}_0) + \pi i[\delta(\mathbf{p}' \cdot \mathbf{v}_0 + \mu'') + \delta(\mu'')] \\ & \times P \frac{1}{\mathbf{p}' \cdot \mathbf{v}_0} + PP \frac{1}{\mu''(\mathbf{p}' \cdot \mathbf{v}_0 + \mu'')} \end{aligned} \tag{57}$$

Associated with the first double delta function member of (57) is, from (55), the combination of dc conductivities

$$\operatorname{Re}[\hat{\sigma}_Q(\mathbf{p}', \mathbf{p}''; 0, 0) - \hat{\sigma}_Q(\mathbf{p}', -\mathbf{p}' - \mathbf{p}''; 0, 0) - \hat{\sigma}_Q(-\mathbf{p}' - \mathbf{p}'', \mathbf{p}''; 0, 0)]$$

each of which is identically zero in virtue of (53b) and the boundedness of the corresponding dc terms in Im η.[6] Thus the first member of (57) contributes nothing to $S^{(3)}$ in (55). We might observe from the odd parity of Re $\hat{\sigma}_Q$

[6] From our recent nonlinear fluctuation-dissipation theorem study[9] we were able to show that

$$\operatorname{Im} \eta(\mathbf{p}', \mathbf{p}''; 0, 0) = (-\beta^2/2\epsilon_0 L^3 p p' p'') \langle \rho(\mathbf{p}) \rho(-\mathbf{p}') \rho(-\mathbf{p}'') \rangle^{(0)}, \qquad \mathbf{p} = \mathbf{p}' + \mathbf{p}''$$

where, physically, one expects that the r.h.s. equal-time triplet correlation of microscopic charge densities is bounded.

with respect to simultaneous sign reversal of its wave vector and frequency arguments[7] that the second and third mixed delta function-principal part members of (57) also contribute nothing to the power loss expression (44) due to the fact that these two members, each in multiplication with the other wave-vector- and frequency-dependent factors comprising (44), give rise to a net oddness parity. Thus, only the fourth pure double principal part member of (57) can contribute and, fortunately, this term always carries the same sign independent of the four possible choices given in (56). Thus a partial fraction expansion of the resulting triple principal part fraction in (55) accompanied by applications of Kramers–Kronig formulas like

$$\operatorname{Im} \hat{\sigma}(\mathbf{p}', \mathbf{p}''; \mu', \mu'') = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{dx}{\mu'' - x} \operatorname{Re} \hat{\sigma}(\mathbf{p}', \mathbf{p}''; \mu', x) \tag{58}$$

and subsequent conversion from $\hat{\sigma}$ to η via (53b) ultimately yields for the triplet form factor

$$\begin{aligned} S^{(3)}(\mathbf{p}', \mathbf{p}'') = \frac{8\pi^2\epsilon_0 L^3}{\beta^2} \Big[& -\frac{1}{\mathbf{p}'' \cdot \mathbf{v}_0} \operatorname{Re} \eta(\mathbf{p}', \mathbf{p}''; \mathbf{p}' \cdot \mathbf{v}_0, 0) \\ & + \frac{1}{\mathbf{p}'' \cdot \mathbf{v}_0} \operatorname{Re} \eta(\mathbf{p}', \mathbf{p}''; \mathbf{p}' \cdot \mathbf{v}_0, \mathbf{p}'' \cdot \mathbf{v}_0) \\ & - \frac{1}{(\mathbf{p}' + \mathbf{p}'') \cdot \mathbf{v}_0} \operatorname{Re} \eta(\mathbf{p}', -\mathbf{p}' - \mathbf{p}''; \mathbf{p}' \cdot \mathbf{v}_0, -(\mathbf{p}' + \mathbf{p}'') \cdot \mathbf{v}_0) \\ & + \frac{1}{(\mathbf{p}' + \mathbf{p}'') \cdot \mathbf{v}_0} \operatorname{Re} \eta(\mathbf{p}', -\mathbf{p}' - \mathbf{p}''; \mathbf{p}' \cdot \mathbf{v}_0, 0) \\ & + \frac{1}{\pi} PPP \int_{-\infty}^{\infty} \frac{d\mu''}{\mu''} \frac{\mathbf{p}' \cdot \mathbf{v}_0}{(\mathbf{p}'' \cdot \mathbf{v}_0 - \mu'')(\mathbf{p}' \cdot \mathbf{v}_0 + \mu'')} \\ & \times \operatorname{Im} \eta(-\mathbf{p}' - \mathbf{p}'', \mathbf{p}''; \mu'' + \mathbf{p}' \cdot \mathbf{v}_0, -\mu'') \Big] \end{aligned} \tag{59}$$

6. DIELECTRIC FORMULATION OF THE STATISTICAL POWER LOSS AND THE PLASMA EXPANSION PARAMETER

Kalman's linear contribution to the statistical power loss,[5]

$$P_{\text{STAT}}^{(2)} = \frac{(Ze)^2}{\beta m_0 \epsilon_0 L^3} \sum_{\mathbf{p}} \left(1 + \frac{\mathbf{p} \cdot \mathbf{v}_0}{p^2} \mathbf{p} \cdot \frac{\partial}{\partial \mathbf{v}_0} \right) \frac{1}{\mathbf{p} \cdot \mathbf{v}_0} \operatorname{Im} \frac{1}{\epsilon_L(\mathbf{p}, \mathbf{p} \cdot \mathbf{v}_0)} \tag{60}$$

is readily obtained from the combination of (43), (45), (47), (53a), and (54). Unlike the polarization power loss (29a), this statistical contribution is, of course, proportional to the plasma temperature β^{-1}. Equation (60) also

[7] The parity rules for $\hat{\sigma}_Q$ are the same as those for η in footnote 5.

shows that more massive test particles are less affected by other statistical fluctuations of the equilibrium plasma.

Concerning the corresponding hihger-order contribution and, in particular, the triplet form factor, we observe that the wave vector transformations

$$\mathbf{p}' = \mathbf{q}'' - \mathbf{q}', \qquad \mathbf{p}'' = \mathbf{q}', \qquad \mathbf{p}' + \mathbf{p}'' = \mathbf{q}''$$

put $S^{(3)}$ into a form which is *manifestly antisymmetric* under interchange of $\mathbf{q}'$ and $\mathbf{q}''$, i.e.,

$$Z(\mathbf{q}', \mathbf{q}'') = S^{(3)}(\mathbf{q}'' - \mathbf{q}', \mathbf{q}') = -S^{(3)}(\mathbf{q}' - \mathbf{q}'', \mathbf{q}'') = -Z(\mathbf{q}'', \mathbf{q}') \tag{61}$$

This parity rule permits us to write the power loss equation (44) in the manifestly (prime, double prime interchange) symmetric form:

$$\begin{aligned} P^{(3)}_{\text{POL-STAT+STAT}} = {} & -\frac{(Ze)^3}{(\beta m_0 \epsilon_0 L^3)^2} \sum_{\mathbf{q}', \mathbf{q}''} \frac{(\mathbf{q}' \cdot \mathbf{q}'')(\mathbf{q}'' - \mathbf{q}')_\alpha (\mathbf{q}'' - \mathbf{q}')_\mu (q_\beta' q_\alpha'' - q_\beta'' q_\alpha')}{|\mathbf{q}'' - \mathbf{q}'|^2 (q' q'')^3 [1 - (\hat{q}' \cdot \hat{q}'')^2]^2} \\ & \times \Big\{ \beta m_0 v_{0\mu} v_{0\nu} [\mathbf{q}'' \cdot \mathsf{T}(\hat{q}') \cdot \hat{q}'' \hat{q}_\nu'' - \mathbf{q}' \cdot \mathsf{T}(\hat{q}'') \cdot \hat{q}' \hat{q}_\nu'] \frac{\partial}{\partial v_{0\beta}} \\ & - \left[\frac{\hat{q}_\mu'' \hat{q}_\lambda'' T_{\gamma\lambda}(\hat{q}') q_\nu'}{q''} - \frac{\hat{q}_\mu' \hat{q}_\lambda' T_{\gamma\lambda}(\hat{q}'') q_\nu''}{q'} \right] v_{0\nu} \frac{\partial^2}{\partial v_{0\beta} \partial v_{0\gamma}} \Big\} \\ & \times \frac{R^{(3)}(\mathbf{q}', \mathbf{q}'')}{(\mathbf{q}'' - \mathbf{q}') \cdot \mathbf{v}_0} \end{aligned} \tag{62}$$

where

$$R^{(3)}(\mathbf{q}', \mathbf{q}'') = (-\beta^2/8\pi^2 \epsilon_0 L^3)\, Z(\mathbf{q}', \mathbf{q}'')$$

From (59) and (61) we see that $R^{(3)}(\mathbf{q}', \mathbf{q}'')$ is evidently an "effective" quadratic polarizability-like response function. Again remark that the term in (62) which carries the second velocity derivative is the pure statistical contribution. This term is seen to be proportional to the square of the temperature, consistent with the fact that it is a quadratic correction to the linear statistical power loss. On the other hand, the term which carries the first velocity derivative is the mixed polarization-statistical effect and therefore features only the first power of the temperature. Equation (62) is the desired dielectric formulation which extends Kalman's result (6?).

Equations (29a) and (60) show that dissipation is portrayed by Im ϵ_L. This well-known result has been deduced from purely medium electrodynamic considerations.[11] Concerning the quadratic contribution to power absorption, we observe from (29b) and (62) that for a zero-temperature and, consequently, nondissipative plasma, η must be purely imaginary. This result has been corroborated in a recent nonlinear conductivity calculation based on a cold plasma hydrodynamic model.[12] However, it would be preferable to verify from a model-independent medium electrodynamic energy absorption calculation that η is indeed purely imaginary for a nondissipative plasma.

Unlike a linear medium, where absorption is reflected entirely by Im ϵ_L, the quadratic plasma medium features both the real and imaginary parts of η [see Eq. (59)]. Assuredly the real part seems to be more prominent and this seems all the more true, since the appearance of Im η under an integral, e.g.,

$$P \int_{-\infty}^{\infty} (d\mu''/\mu'') \, \text{Im}\, \eta(-\mathbf{p}' - \mathbf{p}'', \mathbf{p}''; \mu' + \mathbf{p}' \cdot \mathbf{v}_0, -\mu'')$$

arises only because the variable of integration (in this case, μ'') appears in each of the two frequency arguments, so that Kramers-Kronig formulas cannot be used to any advantage here.

The derivation of the purely dielectric description (62) for the quadratic correction to test particle power loss has been the central thrust of this paper. In principle, it is then a straightforward matter to compute the dielectric functions in (62) from the appropriate plasma kinetic equation to obtain explicit values for the various contributions (polarization, statistical, and mixed) to this power loss correction as a function of the ratio of the test charge velocity to the thermal velocity of the equilibrium plasma. For the linear contribution Kalman[(6)] has already evaluated (60) to zeroth order in the coupling parameter (ratio of potential energy to kinetic energy), $\delta = \beta e^2/4\pi\epsilon_0 r$, using the *Vlasov* expression for the linear polarizability therein. Similar computations applied to (62) are much more complicated and in any case fall outside of the scope of the present paper.

We close this study with a proper clarification of the ordering of the smallness parameters δ and $\gamma = \beta n e^2 r^2/\epsilon_0$ when the quadratic corrections are taken into account. We remind the reader that the density n is to be regarded as being a parameter independent of the coupling strength e^2; thus n may be large even in the weak coupling limit, so that the smallness of the *plasma* parameter,

$$3/4\pi n L_d^3 = 3\gamma^{1/2}\delta$$

can be maintained by taking $\delta \ll 1$ for $\gamma \gg 1$. The smallness of δ is, in turn, assured by excluding fluctuations which arise from strong binary collisions, i.e., excluding interaction distances r which are smaller than the impact parameter. Thus our perturbation scheme is valid if, and only if,

$$\beta e^2 p_l/4\pi\epsilon_0 \ll 1$$

To determine the ordering of γ and δ in (62), let us consider, for example, the nonresonant ($\epsilon_L \simeq 1$) contribution. To lowest order in the charge smallness (i.e., in the Vlasov approximation), one has from the Kalman-Pomeau study[13] that

$$\operatorname{Re} \eta(\mathbf{k}', \mathbf{k}''; \omega', \omega'') = -\frac{\pi\beta n e^2 e}{m\epsilon_0 k k' k''}$$
$$\times \left\{ \int d^3\mathbf{v}\, \delta(\omega - \mathbf{k}\cdot\mathbf{v}) \left(\mathbf{k}' \cdot \frac{\partial}{\partial \mathbf{v}} P \frac{f^0}{\omega'' - \mathbf{k}''\cdot\mathbf{v}} + \mathbf{k}'' \cdot \frac{\partial}{\partial \mathbf{v}} P \frac{1}{\omega' - \mathbf{k}'\mathbf{v}} \right) \right.$$
$$+ \int d^3\mathbf{v}\, P \frac{1}{\omega - \mathbf{k}\cdot\mathbf{v}}$$
$$\left. \times \left[\mathbf{k}' \cdot \frac{\partial}{\partial \mathbf{v}} f^0\, \delta(\omega'' - \mathbf{k}\cdot\mathbf{v}) + \mathbf{k}'' \cdot \frac{\partial}{\partial \mathbf{v}} f^0\, \delta(\omega' - \mathbf{k}'\cdot\mathbf{v}) \right] \right\} \qquad (63)$$

where $\mathbf{k} = \mathbf{k}' + \mathbf{k}''$ and f^0 is the Maxwellian distribution normalized to unity. Thus to lowest order, $\operatorname{Re}\eta \sim \gamma e$, so that

$$P^{(3)}_{\text{Vlasov}} \sim (e^3)(\gamma e) = O(\gamma\delta^2)$$

For the linear power loss expression one finds in the Vlasov approximation that

$$P^{(2)}_{\text{Vlasov}} = O(\gamma\delta)$$

Clearly, then, if the quadratic correction $P^{(3)}$ is to be included in the total power loss P, then P can be correct to $O(\gamma\delta^2)$ only if $P^{(2)}$ includes collisional corrections which are of order δ, i.e.,

$$P^{(2)}_{\text{Vlasov}+\text{collisions}} = O(\gamma\delta)(1 + \delta)$$

Thus the linear polarizability of (59) should be evaluated from a plasma kinetic equation for the one-particle distribution function which displays a collision operator on its right-hand side.

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